BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 3

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CONTENTS

I. REVIEW

Last time we:

(1) Defined affine plane curves as vanishing sets in $\mathbb{A}^2$ of irreducible polynomials, and projective plane curves as vanishing sets in $\mathbb{P}^2$ of irreducible *homogeneous* polynomials.

(2) Showed that a plane curve can be given the structure of a Riemann surface. More precisely,

Proposition 1. *Let $X : F(x_0, x_1, x_2) = 0$ be a nonsingular projective plane curve, where $F \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous. Then X is a compact, connected Riemann surface. Moreover, at every point of X one can take a ratio of the homogeneous coordinates as a local coordinate.*

II. EXAMPLES OF PROJECTIVE PLANE CURVES

Example 2.

(1) (Elliptic curves) An elliptic curve over a field k is a smooth projective plane curve given by an equation of the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

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with $a_i \in k$. Such an equation is called a Weierstrass equation. Over a field of characteristic $\neq 2$ or 3 (like $\mathbb{C}$, for instance), one can make a change of variable and obtain a short Weierstrass equation of the form

$$Y^2Z = X^3 + aXZ^2 + bZ^3 .$$

This is the closure in $\mathbb{P}^2$ of the affine elliptic curve given by $y^2 = x^3 + ax + b$, where we embed $\mathbb{A}^2$ in $\mathbb{P}^2$ as the standard affine open U_2 where $Z \neq 0$. A curve given by a short Weierstrass equation as above is smooth iff $4a^3 + 27b^2 \neq 0$. (This is the negative of the discriminant of the cubic $x^3 + ax + b$.)

(2) (Fermat curves) A Fermat curve is a projective plane curve given by an equation of the form

$$X^d + Y^d = Z^d$$

for some $d \in \mathbb{Z}_{\geq 1}$. Again it is the closure of the affine Fermat curve $x^d + y^d = 1$ in $\mathbb{P}^2$.

III. More complex analysis: singularities and Laurent series

III.1. Laurent series.

Definition 3. Fix $z_0 \in \mathbb{C}$. A Laurent series centered at z_0 is a doubly infinite series of the form

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

where $a_n \in \mathbb{C}$ for all $n \in \mathbb{Z}$.

A Laurent series converges on an annulus, i.e., a region of the form

$$D = \{z \in \mathbb{C} : \rho_I < |z - z_0| < \rho_O\}$$

for some nonnegative real numbers $\rho_I < \rho_O$. (As with Taylor series, it may also converge on subsets of the inner or outer boundary.) Note that in the extreme case where $\rho_I = 0$, the set is a punctured disc

$$D = D^*(z_0, \rho_O) = \{z \in \mathbb{C} : 0 < |z - z_0| < \rho_O\}$$

and when $\rho_O = \infty$, the set is the complement of a disc:

$$D = \mathbb{C} \setminus \overline{D(z_0, \rho_I)} = \{z \in \mathbb{C} : \rho_I < |z - z_0|\} .$$

Theorem 4. *Suppose that f is holomorphic on an annulus*

$$D = \{z \in \mathbb{C} : a < |z - z_0| < b\}$$

where $0 \leq a < b \leq \infty$. Then f can be represented by a Laurent series on D, i.e, there exist coefficients a_n such that

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$$

for all $z \in D$. Moreover, this representation is unique: the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

for any $r \in \mathbb{R}$ with $a < r < b$.

Example 5.

(1) Since $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z \in \mathbb{C}$, then $e^{1/z}$ is holomorphic on $D = \mathbb{C} \setminus \{0\}$ and has Laurent series

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = 1 + \frac{1}{z} + \frac{1}{2! z^2} + \cdots$$

or equivalently,

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{z^n}{(-n)!} .$$

(2) $f(z) = 1/z$ is holomorphic on the annulus $D = \{z \in \mathbb{C} : 1 < |z-1|\}$ and has Laurent series

$$\frac{1}{z} = \frac{1}{1 + (z-1)} = \frac{1}{z-1} \, \frac{1}{\frac{1}{z-1} + 1}$$

Since $1 < |z-1| \iff \frac{1}{|z-1|} < 1$, then we can expand this as a geometric series:

$$\frac{1}{z} = \frac{1}{z-1} \, \frac{1}{1 + \frac{1}{z-1}} = \frac{1}{z-1} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z-1} \right)^n = \frac{1}{z-1} \sum_{n=-\infty}^{0} (-1)^n (z-1)^n$$

$$= \sum_{n=-\infty}^{0} (-1)^n (z-1)^{n-1} = \sum_{n=-\infty}^{-1} (-1)^{n+1} (z-1)^n = \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \cdots .$$

III.2. **Singularities and meromorphic functions.** There are 3 types of singularities that can occur: removable singularities, poles, and essential singularities.

Fix $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$ and suppose that f is holomorphic on the punctured disc $D^* := D^*(z_0, r)$, but is not differentiable at z_0. Then f can be represented uniquely as a Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ on D^*.

- (Removable). If $a_n = 0$ for all $n < 0$, then f has a removable singularity at z_0. Thus its Laurent series is really just a Taylor series. It's removable in that, if we simply redefine $f(z_0) = a_0$, then f becomes differentiable at z_0 and hence analytic on the whole disc $D(z_0, r)$.
- (Pole). If $a_n \neq 0$ for at least one, but only finitely many $n < 0$, then f has a pole at z_0. Then its Laurent series has only a finite tail on the lefthand side. Thus there exists a positive integer m such that $a_{-m} \neq 0$ but $a_{-n} = 0$ for all $n > m$, so the Laurent series is of the form

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots \frac{a_{-1}}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n .$$

  With this notation, we say that f has a pole of order m at z_0.
- (Essential). If $a_n \neq 0$ for infinitely many $n < 0$, then f has an essential singularity at z_0. The behavior of a function near an essential singularity is wild!

Theorem 6 (Casorati-Weierstrass). *Suppose f is analytic on the punctured disc $D^* := D^*(z_0, r)$ and has an essential singularity at z_0. Then $f(D^*)$ is dense in $\mathbb{C}$.*

Theorem 7 (Picard's Little Theorem). *Suppose f is analytic on the punctured disc $D^* := D^*(z_0, r)$ and has an essential singularity at z_0. Then $f(D^*)$ is either all of $\mathbb{C}$, or $\mathbb{C}$ minus a single point.*

Remark 8. There's even Picard's Great Theorem, which states that f takes on all these values infinitely often!

Definition 9. The singular part or principal part of f at z_0 is the Laurent tail consisting of all terms with negative powers:

$$S(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

Example 10. z/z has a removable singularity at $z = 0$, $1/z$ has a simple pole at $z = 0$, and $e^{1/z}$ has an essential singularity at $z = 0$.

Definition 11. Let $U \subseteq \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is meromorphic on U if f has at no point of U worse than a pole, i.e., no essential singularities.

Proposition 12. *Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be meromorphic. Then there exist holomorphic functions $g, h : U \to \mathbb{C}$ such that $f = g/h$.*

Remark 13. Let $\mathcal{O}(U)$ and $\mathcal{M}(U)$ be the set of holomorphic and meromorphic functions, respectively, on U. The above proposition shows that $\mathrm{Frac}(\mathcal{O}(U)) = \mathcal{M}(U)$. (Technically only one implication, but the other is easier.)

Theorem 14 (Discreteness of zeroes and poles). *Let $U \subseteq \mathbb{C}$ be a domain and $f : U \to \mathbb{C}$ be a nonconstant meromorphic function. Then the sets of zeroes and poles of f are discrete subsets of U.*

Proof. If the sets of zeroes of f had a limit point, then f would be constant by the Identity Theorem, contradiction. □

IV. MORPHISMS OF RIEMANN SURFACES

IV.1. **Definition and first examples.** We now extend the idea of holomorphicity to Riemann surfaces by defining it locally using coordinate charts.

Definition 15. Let X be a Riemann surface. A function $f : X \to \mathbb{C}$ is holomorphic (resp., meromorphic) if for any coordinate map $\varphi : U \to \widehat{U} \subseteq \mathbb{C}$, the function $f \circ \varphi^{-1} : \widehat{U} \to \mathbb{C}$ is holomorphic (resp., meromorphic).

Given $U \subseteq X$ open, let

$$\mathcal{O}_X(U) := \{f : U \to \mathbb{C} \mid f \text{ is holomorphic}\}$$
$$\mathcal{M}_X(U) := \{f : U \to \mathbb{C} \mid f \text{ is meromorphic}\}.$$

The set of all meromorphic functions $X \to \mathbb{C}$ is a field called the function field and is denoted $\mathcal{M}(X)$ or $\mathbb{C}(X)$.

Definition 16. A morphism or holomorphic map between Riemann surfaces X_1 and X_2 is a continuous map $f : X_1 \to X_2$ such that $\psi \circ f \circ \varphi^{-1}$ is holomorphic for any choice of coordinate φ on X_1 and ψ on X_2 for which the composition is defined. The set of all morphisms $X_1 \to X_2$ is denoted by $\mathrm{Mor}(X_1, X_2)$.

A holomorphic map with a holomorphic inverse is an isomorphism of Riemann surfaces, and an isomorphism from a Riemann surface X to itself is an automorphism. The set of automorphisms of X forms a group, denoted by $\mathrm{Aut}(X)$.

Example 17. Here are some examples of morphisms of Riemann surfaces.

(1) Given $n \in \mathbb{Z}_{\geq 0}$, define $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$f(z) = \begin{cases} z^n & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty \,. \end{cases}$$

(2) Given $n \in \mathbb{Z}_{\geq 0}$, define

$$\begin{aligned} f : \mathbb{P}^1 &\to \mathbb{P}^1 \\ [X : Y] &\mapsto [X^n : Y^n] \,. \end{aligned}$$

(3) Let $E : Y^2 Z = X^3 + Z^3$ and define

$$\begin{aligned} f : E &\to \mathbb{P}^1 \\ [X : Y : Z] &\mapsto [X : Z] \,. \end{aligned}$$

(4) Similarly, let $C : X^3 + Y^3 = Z^3$ and define

$$\begin{aligned} f : C &\to \mathbb{P}^1 \\ [X : Y : Z] &\mapsto [X : Z] \,. \end{aligned}$$

Example 18.

(1) $\mathfrak{H}$ and $\mathfrak{D}$ are isomorphic as Riemann surfaces. Such an isomorphism is given by

$$\begin{aligned} \mathfrak{H} &\to \mathfrak{D} \\ z &\mapsto \frac{z-i}{z+i} \,. \end{aligned}$$

(2) $\mathbb{C}$ and $\mathfrak{D}$ are not isomorphic. In fact $\mathrm{Mor}(\mathbb{C}, \mathfrak{D})$ consists only of the constant maps.

(3) $\mathbb{S}^2$, $\mathbb{P}^1$, and $\widehat{\mathbb{C}}$ are isomorphic via the maps

$$\begin{aligned} \mathbb{P}^1 &\to \widehat{\mathbb{C}} \\ [z_0 : z_1] &\mapsto z_1 / z_0 \\ [0 : 1] &\mapsto \infty \end{aligned}$$

$$\begin{aligned} \mathbb{S}^2 &\to \widehat{\mathbb{C}} \\ (x, y, t) &\mapsto \frac{x + iy}{1 - t} \\ (0, 0, 1) &\mapsto \infty \,. \end{aligned}$$

IV.2. **Fundamental results on morphisms.**

Proposition 19. *Let X and Y be compact, connected Riemann surfaces. Then every nonconstant morphism $f : X \to Y$ is surjective.*

Proof. By the open mapping theorem, $f(X)$ is open. And since $f(X)$ is compact and Y is Hausdorff, then $f(X)$ is also closed. Since Y is connected, then $f(X) = Y$. □

Proposition 20. *Let X be a Riemann surface and denote by $c_P \in \operatorname{Mor}(X, \widehat{\mathbb{C}})$ the constant morphism $c_P(x) = P$ for all x. Then*

$$\mathcal{M}(X) = \operatorname{Mor}(X, \widehat{\mathbb{C}}) \setminus \{c_\infty\}.$$

Proposition 21. *$\mathcal{M}(\widehat{\mathbb{C}}) = \mathbb{C}(z)$, the field of rational functions in one variable.*

Proposition 22. *Let X be a compact, connected Riemann surface. Then every holomorphic function $f : X \to \mathbb{C}$ is a constant map.*

Proof. Composing with the inclusion $\iota : \mathbb{C} \to \widehat{\mathbb{C}}$, we obtain the morphism $\iota \circ f : X \to \widehat{\mathbb{C}}$. Since nonconstant maps of compact, connected Riemann surfaces are surjective and $\infty \notin \operatorname{img}(\iota \circ f)$, then $\iota \circ f$, and hence f, must be constant. □

Proposition 23. *The automorphism groups of $\mathbb{P}^1$ and $\mathbb{C}$ are:*

$$\begin{aligned} \operatorname{Aut}(\mathbb{P}^1) &= \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C}, ad-bc \neq 0 \right\} \\ &\cong \mathrm{PGL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C}) \\ \operatorname{Aut}(\mathbb{C}) &= \{ z \mapsto az+b : a,b \in \mathbb{C} \} . \end{aligned}$$

Proof. Let f be a meromorphic function on $\mathbb{P}^1$. Identifying $\mathbb{P}^1$ and $\widehat{\mathbb{C}}$ by the isomorphism given above, then we can consider f as a meromorphic function on $\widehat{\mathbb{C}}$. By Proposition 21, then f is a rational function in z, hence can be written

$$f(z) = \lambda \frac{(z-b_1)\cdots(z-b_m)}{(z-a_1)\cdots(z-a_n)} .$$

In order for f to be bijective, both the numerator and denominator must have degree ≤ 1. Thus

$$f(z) = \frac{az+b}{cz+d}$$

for some a, b, c, d. Moreover, since f is invertible, one can show that $ad - bc \neq 0$. A straightforward computation shows that composition of Möbius transformations corresponds to matrix multiplication, so there is a group homomorphism

$$\begin{aligned} \varphi : \mathrm{GL}_2(\mathbb{C}) &\to \operatorname{Aut}(\mathbb{P}^1) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \left(z \mapsto \frac{az+b}{cz+d} \right) . \end{aligned}$$

The kernel of φ consists of the center

$$Z(\mathrm{GL}_2(\mathbb{C})) = \left\{ \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix} : \mu \in \mathbb{C}^\times \right\}$$

Since φ is surjective, we have

$$\mathrm{PGL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})/Z(\mathrm{GL}_2(\mathbb{C})) \cong \mathrm{Aut}(\mathbb{P}^1)\,.$$

Given $M \in \mathrm{GL}_2(\mathbb{C})$, then

$$\begin{pmatrix} \frac{1}{\sqrt{\det(M)}} & 0 \\ 0 & \frac{1}{\sqrt{\det(M)}} \end{pmatrix} M$$

defines the same element of $\mathrm{PGL}_2(\mathbb{C})$ and has determinant 1. This shows that $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{PSL}_2(\mathbb{C})$, where

$$\mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/Z(\mathrm{SL}_2(\mathbb{C})) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}\,.$$

Let $f \in \mathrm{Aut}(\mathbb{C})$. We claim that f extends to an automorphism of $\widehat{\mathbb{C}}$. It suffices to show that f cannot have an essential singularity at ∞. If f has an essential singularity at ∞, then $f(\{|z| > 1\})$ is dense in $\mathbb{C}$. But since f is injective, then $f(\{|z| < 1\})$ and $f(\{|z| > 1\})$ are disjoint, contradicting denseness of the latter.

Thus f extends to an automorphism of $\widehat{\mathbb{C}}$, so f is of the form

$$f(z) = \frac{az+b}{cz+d}\,.$$

But since $f(\mathbb{C}) \subseteq \mathbb{C}$, then f has no poles in $\mathbb{C}$, so $c = 0$. □

IV.3. **Order of vanishing and multiplicity.**

Definition 24. Let X be a Riemann surface, $P \in X$, and $f \in \mathcal{M}(X)$ be a meromorphic function. Let φ be a centered coordinate map at P, so $\varphi(P) = 0$. Then f can represented by the Laurent series $f \circ \varphi^{-1}(z) = \sum_n a_n z^n$. The order (of vanishing) of f at P, denoted by $\mathrm{ord}_P(f)$ is the smalles n such that $a_n \neq 0$:

$$\mathrm{ord}_P(f) := \min\{n \in \mathbb{Z} : a_n \neq 0\}\,.$$

If $\mathrm{ord}_P(f)n \geq 1$, then f has a zero of order n at P and if $\mathrm{ord}_P(f) = -n < 0$, then f has a pole of order n at P.

Remark 25. One can show that the order is independent of the choice of coordinate chart.

Lemma 1. *Let $f, g \in \mathcal{M}(X)$ be meromorphic functions on a Riemann surface X. Then*

(1) $\mathrm{ord}_P(fg) = \mathrm{ord}_P(f) + \mathrm{ord}_P(g)$;
(2) $\mathrm{ord}_P(1/f) = -\mathrm{ord}_P(f)$;
(3) $\mathrm{ord}_P(f+g) \geq \min\{\mathrm{ord}_P(f), \mathrm{ord}_P(g)\}$.

Remark 26. This shows that ord_P is a discrete valuation on $\mathcal{M}(X)$ for each point P.

Definition 27. Let $f : X \to Y$ be a morphism of Riemann surfaces, $P \in X$. Let ψ be a chart of Y centered at $f(P)$, so $\psi(f(P)) = 0$. Then

$$m_P(f) := \mathrm{ord}_P(\psi \circ f)$$

is the multiplicity of f at P. Equivalently,

$$m_P(f) = 1 + \mathrm{ord}_P(\psi \circ f)'$$

whether ψ is a centered chart or not.

If $m_P(f) \geq 2$, then $P \in X$ is ramification point or branch point of f, with ramification index $m_P(f)$. A branch value is the image of a ramification point. Equivalently, we say that f is ramified above $Q \in Y$ if there is some $P \in f^{-1}(Q)$ with $m_P(f) \geq 2$ and f is ramified at $P \in X$ if $P \in X$ and $m_P(f) \geq 2$.